# Determination of Extreme Entry Angles into a Planetary Atmosphere

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**Aerodynamic braking may be used as one means of slowing down a vehicle and thus minimizing retrorocket requirements This raises the following problem: what are the steepest and the shallowest angles at which a spaceship may enter a given atmosphere with a given velocity $v_0$ and, after appropriate maneuvering, emerge from the atmosphere at an angle and with a velocity that satisfy certain preassigned conditions? Methods developed previously are applied in this paper to the foregoing problem It is assumed that the planet is a perfect homogeneous sphere and that the motion of the vehicle is confined to a plane passing through the center of the planet The maneuvering consists in modulating the lift-to-drag ratio while keeping its absolute value below a prescribed maximum The problem is reduced to a variational problem of the type discussed in a previous paper One of the conclusions reached is that the optimized lift-to-drag ratio takes on no other values than its permissible minimum and maximum except during the portion of the flight (if such portion exists) when that path is vertical, at which time the lift-to-drag ratio is zero This conclusion was reached assuming that the lift-to-drag ratio enters the equations of motion linearly and that the drag coefficient is constant A computational procedure is developed to determine the thrust-augmented aerodynamic entry corridors and the corresponding lift-to-drag programs and resultant trajectories This procedure is then applied to the study of the thrust-augmented entry corridors for Mars for various vehicle characteristics and conditions**

## Nomenclature

$\gamma$ = angle made by velocity vector of spaceship with local horizontal
$h$ = altitude of spaceship above surface of planet
$v$ = velocity of spaceship
$\rho(h)$ = atmospheric density at altitude $h$
$R$ = radius of planet
$A$ = reference area
$m$ = mass of spaceship
$C_D$ = drag coefficient of spaceship
$\theta(t)$ = ratio $C_L/C_D$ of lift to drag at time $t$
$g$ = acceleration due to gravity at surface of planet

## I Introduction

LET it be assumed that a spaceship is approaching a planet at a high velocity $v_0$ and that it is desired to slow it down substantially (e g , preparatory to placing it into a preassigned orbit about the planet) using aerodynamic braking as one means of saving on the use of retrorockets The ship would thus be made to enter the atmosphere of the planet at a certain angle, would perform a maneuver while in the atmosphere consisting of modulating its lift coefficient within preassigned bounds, and would emerge from the atmosphere at an angle and with a velocity that may have to satisfy certain preassigned conditions (such as, e g , would permit it to achieve a desired orbit using only the available rocket capacity) The question then arises: what are the steepest and shallowest angles at which the spaceship may enter the atmosphere with a given velocity $v_0$ and, after appropriate maneuvering, emerge out of the atmosphere at an angle and with a velocity that satisfy the preassigned conditions, if any? These two angles (the steepest and the shallowest) will be referred to as "extreme entry angles"

Let it be assumed that the planet is a perfect sphere and that the motion of the spaceship is confined to a plane passing through the center of the planet The equations of motion of the spaceship are then

$$\frac{d\gamma}{dt} = \frac{v}{R+h}\cos\gamma + \frac{1}{2}\frac{A}{m}\rho(h)vC_D\theta(t) - \frac{g}{v}\left(\frac{R}{R+h}\right)^2\cos\gamma$$

$$\frac{dh}{dt} = v\sin\gamma \qquad (1)$$

$$\frac{dv}{dt} = -\frac{1}{2}\frac{A}{m}\rho(h)v^2C_D - g\left(\frac{R}{R+h}\right)^2\sin\gamma$$

It is assumed that the atmosphere effectively extends up to the altitude $h_E$, that $C_D$ is fixed, and that $\theta(t)$ can be varied at will as time progresses while remaining bounded by the numbers $-a$ and $a$, where $a$ is a preassigned number

The conditions that are imposed on the velocity $\bar{v}$ and the angle $\bar{\gamma}$ of the vehicle when it emerges from the atmosphere may be rather arbitrary The following four cases will be considered: 1) $\bar{v}$ and $\bar{\gamma}$ are arbitrary; 2) $\phi(\bar{v},\bar{\gamma}) \leq 0$, where $\phi$ is a given differentiable function; 3) $\phi(\bar{v},\bar{\gamma}) = 0$, where $\phi$ is a given differentiable function; and 4) $\bar{v}$ and $\bar{\gamma}$ have preassigned values

The mathematical problem of determining the steepest (respectively shallowest) entry angle can now be described as follows: determine a positive value $T$, a function $\theta(t)$ defined over the interval $0 \leq t \leq T$ such that $-a \leq \theta(t) \leq a$, and values $\bar{v}$ and $\bar{\gamma}$ satisfying cases 1–4, respectively, so that the solution of Eq (1) with "initial conditions" $v(T) = \bar{v}$, $\gamma(T) = \bar{\gamma}$, $h(T) = h_E$ is such that $v(0) = v_0$, $h(0) = h_E$, and $\gamma(0)$ is minimum (respectively maximum)

A procedure for determining the steepest (respectively shallowest) of the extreme angles and the corresponding maneuver will be discussed in the ensuing sections of this paper This procedure is based on the approach of Ref 1

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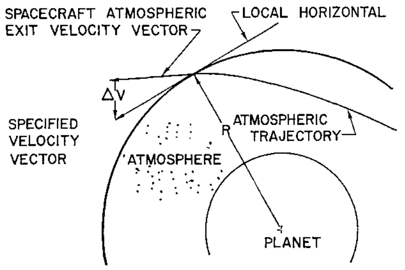


**Fig 1 Exit velocity constrainι**

### Entry Coıιidors into Mars

We shall now briefly describe the results obtained in determining the entry corridor into Mars when the conditions imposed on the vehicle as it emerges from the atmosphere are of a special kind It is intuitively obvious that specifying a unique atmospheric exit velocity vector would impose an unnecessarily severe constraint It was decided, therefore, to permit the spacecraft atmospheric exit velocity $\mathbf{v}_E$ to be such that a horizontal vector velocity of desired magnitude (circular orbit velocity) could be obtained at exit time by adding to $\mathbf{v}_E$ an incremental velocity vector $\mathbf{\Delta v}$ of specified magnitude and appropriate direction (see Fig 1) This condition is of the type of case 3, where the function $\phi(\bar{v},\bar{\gamma}) = \bar{v}^2 - 2v_{ED}\bar{v}\cos\bar{\gamma} + v_E{}^2 - \Delta v^2 = 0$, $\bar{v}$ and $\bar{\gamma}$ are the velocity and angle of the exiting vehicle, $v_E$ is the desired scalar velocity, and $\Delta v = |\mathbf{\Delta v}|$

Figure 2 depicts the thrust-augmented aerodynamic entry corridor for Mars as a function of $m/C_DA$ and $\Delta v$ for entry velocity of 25,000 fps, entry and exit altitude of 800,000 ft, orbit velocity of 11,200 fps, and $a = |L/D|_{m\ x} = 0$ and 0 5 The atmospheric density model employed is presented in Table 1

When $a = |L/D|_{m\ x} =$ 0 5, it is apparent from Fig 2 that the shallowest entry angle (corresponding to the "overshoot" trajectory) is insensitive to $\Delta v$ for all values of $m/C_DA$ being considered On the other hand, the steepest entry angle (corresponding to the "undershoot" trajectory) vaıies substantially with $\Delta v$ for each of the values of $m/C_DA$ under consideration Although it is not shown on the figure, the increase in the corridor width falls off abruptly somewhat beyond the $\Delta v = 3000$ fps bound The reason for this is that an entry angle is approached for which the vehicle can no longer emerge from the atmosphere, i e , max $L/D$ exerted in a positive direction throughout the flight is inadequate to effect a pullout

The depicted corridors represent the theoretical maximum attainable for the stated conditions In other words, if the lift progıam were pıepared with the complete knowledge of all pertinent parameters and conditions and were faithfully executed, these would be the attainable corridors To the extent that the actual capacity of the vehicle to maneuver and the actual atmospheric conditions depart from our assumptions, the corridor width will be correspondingly affected

Figure 3 gives a general picture of typical "overshoot" and "undershoot" trajectories, and Fig 4 depicts the time history of a typical "undershoot" trajectory

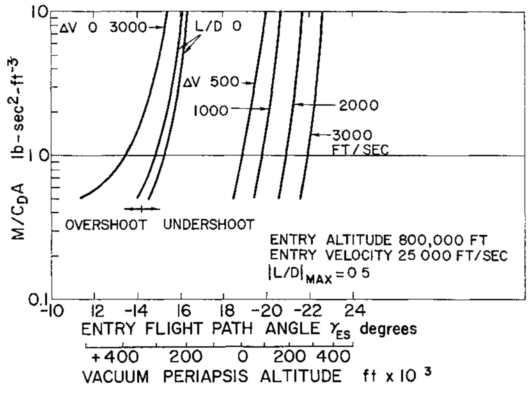


**Fig 2 Thıust-augmented aerodynamic orbit coıridor foı Mars**

**Table 1 Martian atmospheric density**

| Altitude $h$, ft | Density $\rho$, slugs/ft³ |
|---|---|
| 0 0 | 18 991 × $10^{-5}$ |
| 100 000 0 | 4 893 × $10^{-5}$ |
| 200,000 0 | 5 398 × $10^{-6}$ |
| 300 000 0 | 2 486 × $10^{-7}$ |
| 400 000 0 | 8 000 × $10^{-8}$ |
| 500,000 0 | 4 486 × $10^{-8}$ |
| 600 000 0 | 2 796 × $10^{-8}$ |
| 700 000 0 | 1 299 × $10^{-8}$ |
| 800 000 0 | 5 301 × $10^{-9}$ |

## II Necessary Conditions for Minimum

It will be assumed in the sequel that there exists a maneuver that yields the steepest (respectively shallowest) entry angle and that also permits the vehicle to emerge from the atmosphere in a manner consistent with the appropriate condition (case 1, 2, 3, or 4) Furthermore, it will be assumed that the corresponding vehicle path in the atmosphere never touches the surface of the planet and that the vehicle velocity never approaches zero

The existence of such a maneuver should not be always taken for granted If the initial velocity $v_0$, the atmospheric density, or other parameters and functions of the problem are outside appropriate ranges, then it may be that no solution of Eqs (1) can possibly yield "exit" values consistent with cases 1–4, respectively The detailed analysis of conditions that insure the existence of an appropriate maneuver is, however, outside the scope of this paper

Let $x_1 = \gamma$, $x_2 = h$, $x_3 = v$, $x = (x_1,x_2,x_3)$, and let $-f_1(x,\theta)$, $-f_2(x,\theta)$, $-f_3(x,\theta)$ repıesent, respectively, the right-hand sides of the first, second, and third equation of system (1) Setting $\tau = T - t$, $0 \leqq t \leqq T$, $y_i(\tau) = x_i(T - t)$, $i = 1,2,3$, $\sigma(\tau) = \theta(T - t)$, one can now rewrite system (1) in the form

$$dy_i/d\tau = f_i(y,\sigma) \qquad i = 1,2,3 \tag{2}$$

It is desired to consider positive numbers $T$, functions $\sigma(\tau)$ defined over the interval $[0,T]$, and numbers $\bar{v}$ and $\bar{\gamma}$ such that $-a \leq \sigma(\tau) \leq a$ for $0 \leq \tau \leq T$, $\bar{v}$ and $\bar{\gamma}$ satisfy case 1, 2, 3, or 4, and the solution $y_i(\tau)$, $i = 1, 2, 3$ of system (2) with initial conditions

$$y_1(0) = \bar{\gamma} \qquad y_2(0) = h_E \qquad y_3(0) = \bar{v} \tag{3}$$

yields $y_2(T) = h_E$, $y_3(T) = v_0$ Among all such choices of $T$, $\sigma(\tau)$, $\bar{v}$, and $\bar{\gamma}$, it is desired to determine one that yields the minimum (respectively maximum) of $y_1(T)$

It can be easily verified that this problem is a "relaxed variational problem" as defined in Ref 1 Applying the results obtained there and specifically those of theorems 4 1 and 5 1 and the considerations of Sec VII, one obtains the following conditions

There exist functions $z_1(\tau)$, $z_2(\tau)$, $z_3(\tau)$ (not all simultaneously zero) such that

$$\frac{dy_i}{d\tau} = f_i[y(\tau),\sigma(\tau)] \qquad i = 1,2,3 \tag{4}$$

$$\frac{dz_i}{d\tau} = -\sum_{j=1}^{3} \frac{\partial f_j[y(\tau),\sigma(\tau)]}{\partial y_i} z_j \qquad i = 1,2,3 \tag{5}$$

$$\sum_{i=1}^{3} z_i(\tau) f_i[y(\tau),\sigma(\tau)] = \min_{-a \leq \sigma \leq a} \sum^{3} z_i(\tau) f_i[y(\tau),\sigma] = 0 \tag{6}$$

$$z_1(T) \begin{cases} \geq 0 \text{ if we seek the steepest angle} \\ \leq 0 \text{ if we seek the shallowest angle} \end{cases} \tag{7}$$

$$y_2(T) = h_E \qquad y_3(T) = v_0 \tag{8}$$

The conditions at $\tau = 0(t = T)$ depend on which of the restrictions 1–4 is considered Specifically, if one considers case 1, then

$$z_1(0) = 0 \qquad z_3(0) = 0 \tag{9}$$

and if one considers case 2, then either

$$z_1(0) = -\frac{\partial\phi(\bar{v},\bar{\gamma})}{\partial\bar{\gamma}} \qquad z_3(0) = -\frac{\partial\phi(\bar{v},\bar{\gamma})}{\partial\bar{v}} \qquad \phi(\bar{v},\bar{\gamma}) = 0$$

or

$$z_1(0) = 0 \qquad z_3(0) = 0 \qquad \phi(\bar{v},\bar{\gamma}) < 0 \tag{10}$$

and if one considers case 3, then

$$z_1(0) = \epsilon\frac{\partial\phi(\bar{v},\bar{\gamma})}{\partial\bar{\gamma}} \qquad z_3(0) = \epsilon\frac{\partial\phi(\bar{v},\bar{\gamma})}{\partial\bar{v}} \qquad \phi(\bar{v},\bar{\gamma}) = 0 \tag{11}$$

where $\epsilon$ equals either $+1$ or $-1$ If one considers case 4, then

$$\bar{\gamma} = \bar{\gamma} \qquad \bar{v} = \bar{v} \tag{12}$$

where $\bar{\gamma}$ and $\bar{v}$ are the preassigned values

Systems (4) and (5), explicitly written out, are

$$\left.\begin{aligned}
\frac{dy_1}{d\tau} &= -\frac{y_3}{R+y_2}\cos y_1 - \frac{1}{2}\frac{A}{m}\rho(y_2)y_3\sigma C_D + \frac{g}{y_3}\left(\frac{R}{R+y_2}\right)^2\cos y_1 \\
\frac{dy_2}{d\tau} &= -y_3\sin y_1 \\
\frac{dy_3}{d\tau} &= \frac{1}{2}\frac{A}{m}\rho(y_2)y_3^2C_D + g\left(\frac{R}{R+y_2}\right)^2\sin y_1 \\
\frac{dz_1}{d\tau} &= -\left[\frac{y_3}{R+y_2} - \frac{g}{y_3}\left(\frac{R}{R+y_2}\right)^2\right]\sin y_1\ z_1 + y_3\cos y_1\ z_2 - g\left(\frac{R}{R+y_2}\right)^2\cos y_1\ z_3 \\
\frac{dz_2}{d\tau} &= -\left[\frac{y_3}{(R+y_2)^2}\cos y_1 - \frac{1}{2}\frac{A}{m}\frac{d\rho}{dy_2}y_3C_D\sigma - \frac{2gR^2}{y_3(R+y_2)^3}\cos y_1\right]z_1 - \left[\frac{1}{2}\frac{A}{m}\frac{d\rho}{dy_2}y_3^2C_D - \frac{2gR^2}{(R+y_2)^3}\sin y_1\right]z_3 \\
\frac{dz_3}{d\tau} &= \left[\frac{\cos y_1}{R+y_2} + \frac{1}{2}\frac{A}{m}\rho(y_2)C_D\sigma + \frac{g}{y_3^2}\left(\frac{R}{R+y_2}\right)^2\cos y_1\right]z_1 + \sin y_1\ z_2 - \frac{A}{m}\rho(y_2)y_3C_Dz_3
\end{aligned}\right\} \tag{13}$$

Relation (6) implies that $\sigma(\tau)$ minimizes $-z_1(\tau)\sigma$ over the interval $-a \leq \sigma \leq a$; hence,

$$\sigma(\tau) = \begin{cases} a \text{ if } z_1(\tau) > 0 \\ -a \text{ if } z_1(\tau) < 0 \end{cases} \tag{14}$$

This last relation defines $\sigma(\tau)$ uniquely whenever $z_1(\tau) \neq 0$ Now let $Z = \{\tau|0 \leq \tau \leq T, z_1(\tau) = 0\}$ Theorem 5 1 of Ref 1 yields (after a certain amount of algebraic manipulation) the relation

$$y_3\cos y_1\ z_2 - g[R/(R+y_2)]^2\cos y_1\ z_3 = 0 \text{ a e in } Z \tag{15}$$

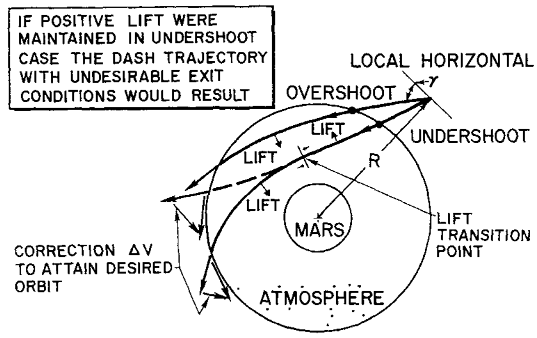


**Fig 3 Entry corridor flight geometry**

Furthermore, relation (6) implies

$$-y_3\sin y_1\ z_2 + \left[\frac{1}{2}\frac{A}{m}\rho(y_2)y_3^2C_D + g\left(\frac{R}{R+y_2}\right)^2\sin y_1\right]z_3 = 0 \text{ a e in } Z \tag{16}$$

Since $z_1 = 0$ in $Z$ and the vector $(z_1, z_2, z_3)$ is nonvanishing, Eqs (15) and (16), considered as linear equations in $z_2$ and $z_3$, must have a singular matrix; hence (after simplification),

$$A\rho y_3^2C_D\cos y_1/2m = 0 \text{ a e in } Z$$

and, since, by assumption, $\rho \neq 0$ and $y_3 \neq 0$, it follows that

$$\cos y_1 = 0 \text{ a e in } Z \tag{17}$$

Since almost every point in a set $Z$ is a limit point of the set, relation (17) implies $dy_1/d\tau = 0$ a e in $Z$; hence, in view of (17) and (13), $A\rho(y_2)y_3C_D\sigma/2m = 0$ a e in $Z$ or $\sigma = 0$ a e in $Z$

## III Computational Approach

It follows from the considerations of Sec II that the functions $y_1(\tau) = \gamma(T-\tau)$, $y_2(\tau) = h(T-\tau)$, $y_3(\tau) = v(T-\tau)$, and the "dual" functions $z_1(\tau)$, $z_2(\tau)$, $z_3(\tau)$ (not all three simultaneously vanishing) satisfy, over the interval $[0,T]$, the simultaneous system of ordinary differential equations (13) in which, almost everywhere in $[0,T]$,

$$\sigma(\tau) = \begin{cases} a \text{ if } z_1(\tau) \geq 0 \text{ and } \cos y_1 \neq 0 \\ -a \text{ if } z_1(\tau) < 0 \\ 0 \text{ if } z_1(\tau) = 0 \text{ and } \cos y_1 = 0 \end{cases} \tag{18}$$

The solutions of this system must satisfy the boundary conditions at $\tau = T$ described in (7) and (8) and the conditions at $\tau = 0$ described in (3 and 9–12), respectively Finally, relation (6) must hold a e in $[0,T]$

In general, the solution of boundary-value problems in ordinary differential equations presents considerable numeri-

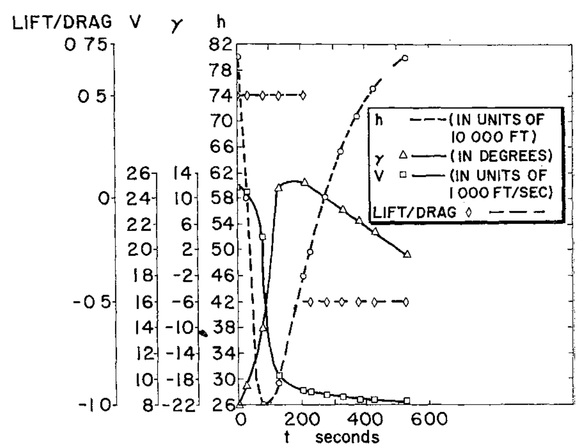


**Fig 4 Trajectory parameters vs time**

cal difficulties However, in all of the cases here considered, the problem can be "usually" solved by varying one single parameter in the initial conditions until one single end condition is satisfied To illustrate the procedure, let case 4 be considered

Since the functions $z_1(\tau)$, $z_2(\tau)$, $z_3(\tau)$ can never vanish simultaneously and since they appear in a linear and homogeneous manner in all the pertinent relations, the latter will remain satisfied if the vector $z(\tau) = [z_1(\tau), z_2(\tau), z_3(\tau)]$ is multiplied by an arbitrary positive constant It may thus be assumed that

$$z_1^2(0) + z_2^2(0) + z_3^2(0) = 1 \tag{19}$$

Now let a value $z_1(0)$ be chosen satisfying $|z_1(0)| \leq 1$ Then relations (18) and (13) permit us to compute $dy_i(0)/dt$ Relation (6) can be rewritten as

$$\sum_{i=1}^{3} z_i(0) \frac{dy_i(0)}{dt} = 0 \tag{20}$$

Thus, for the given $z_1(0)$, (19) and (20) can be solved for $z_2(0)$ and $z_3(0)$, yielding, in general, two solutions: $\bar{z}_2(0)$, $\bar{z}_3(0)$ and $\bar{\bar{z}}_2(0)$, $\bar{\bar{z}}_3(0)$ Each of these initial conditions for $z_1$, $z_2$, $z_3$, combined with $y_1(0) = \bar{\gamma}$, $y_2(0) = h_E$, $y_3(0) = \bar{v}$, permits one to solve system (13) [with $\sigma(\tau)$ defined a e by (18)] for increasing values of $\tau$ If, at some value of $\tau$, one of these solutions yields $y_2(\tau) = h_E$, the value of $y_3(\tau)$ is compared with $v_0$ This process is then repeated for various choices of $z_1(0)$ within the interval $-1 \leq z_1(0) \leq 1$ until one of the corresponding trajectories yields simultaneously, for some value $\tau = T$, $y_3(T) = v_0$ and $y_2(T) = h_E$

This procedure can be followed unless $\cos y_1(\tau) = z_1(\tau) = 0$ for some value of $\tau$ being considered If, for some "trial" solution of system (13) [corresponding to some choice of $z_1(0)$ and either $\bar{z}_2(0)$, $\bar{z}_3(0)$ or $\bar{\bar{z}}_2(0)$, $\bar{\bar{z}}_3(0)$], there exists a value $\tau_1$ such that $\cos y_1(\tau_1) = z_1(\tau_1) = 0$, then a further choice must be made of a number $\Delta\tau_1$ such that $\sigma(\tau)$ is chosen as zero for $\tau_1 < \tau < \tau_1 + \Delta\tau_1$, resulting in the equalities $\cos y_1(\tau) = z_1(\tau) = 0$ holding over the same interval

## IV Determination of a Typical Trajectory

As an illustration, let the following problem be considered A spaceship approaches Mars, and its velocity at an altitude of 800,000 ft over the surface of Mars is 25,000 fps (The Martian atmosphere is assumed to end effectively at 800,000 ft ) It is desired to place the vehicle in a circular orbit about Mars at a height of 800,000 ft by imparting to it at an appropriate time and in an appropriate direction a velocity increment of 3000 fps What is the steepest angle at which the vehicle may enter the atmosphere of Mars so that, with appropriate maneuvering, it may emerge from the atmosphere under conditions permitting one to place it in the desired orbit with the preassigned velocity increment?

The velocity of a Martian satellite following a circular orbit at an elevation of 800,000 ft over the surface of the planet is 11,220 fps If a vehicle emerges from the atmosphere with a velocity $\tilde{v}$ and at an angle $\tilde{\gamma}$ to the local horizontal, an appropriately directed velocity increment of 3000 fps will place it in the desired orbit if

$$3000^2 = 11{,}220^2 + \tilde{v}^2 - 22{,}460\tilde{v} \cos\tilde{\gamma} \tag{21a}$$

or

$$v^2 - 22{,}460\tilde{v} \cos\tilde{\gamma} + 1\ 168886\ 10^8 = 0 \tag{21b}$$

This is a condition of the type specified in case 3 with $\phi(\tilde{v},\tilde{\gamma})$ equal to the left-hand side of Eq (21b) It follows that conditions (11) apply at $\tau = 0$

Now let the following procedure be followed: a value of $\epsilon$ of $+1$ or $-1$ and a value of $\tilde{v}$ are chosen Then Eqs (21) yield a value of $\tilde{\gamma}$ such that $0 < \tilde{\gamma} < 90°$, and relation (11) yields $z_1(0)$ and $z_3(0)$; then (14) yields $\sigma(0)$, and (6) considered at $\tau = 0$, yields $z_2(0)$ Then system (13) [with $\sigma(\tau)$ defined in (14)] is integrated forward with the initial conditions specified by

$$y_1(0) = \tilde{\gamma} \qquad y_2(0) = h_E = 800{,}000 \qquad y_3(0) = \tilde{v}$$

and by the calculated values of $z_1(0)$, $z_2(0)$, $z_3(0)$ If, for some value of $\tau$, $y_2(\tau)$ becomes 0 (the vehicle hits the ground), another choice of $v$ is made and the procedure repeated If, for some value $T$ of $\tau$, $y_2(T) = 800{,}000$, the value of $y_3(T)$ is compared with the number $v_0 = 25{,}000$, and the algebraic sign of $z_1(T)$ is checked

This procedure is repeated for varying values of $v$ and for choices of $\epsilon$ of $+1$ or $-1$ until values are found which yield $y_3(T) = 25{,}000$ and $z_1(T) \geq 0$ Figure 4 represents the time history of $v$, $\gamma$, $h$, and $C_L/C_D$ for a solution obtained by this procedure in the following case:

$$C_D = 0\ 1 \qquad -0\ 5 \leq C_L/C_D \leq 0\ 5 \qquad A/m = 1$$

The following values were also used: $g = 12\ 0534$ ft/sec², $R = 11{,}193{,}600$ ft, and the density table as shown in Table 1

## Reference

[1] Warga, J , "Necessary conditions for minimum in relaxed variational problems," J Math Anal Appl 4(1), 129–145 (1962)